An Ising Spin-S Model on Generalized Recursive Lattice

N.S. Ananikian[*]

Yerevan Physics Institute, Alikhanian Br.2, 375036 Yerevan, Armenia

N.Sh. Izmailian [†‡]

Institute of Physics, Academia Sinica, Taipei, Taiwan 11529, Republic of China

K.A. Oganessyan [§¶]

LNF-INFN, C.P. 13, Enrico Fermi 40, Frascati, Italy

February 1, 2008

Abstract

The Ising spin - S model on recursive p - polygonal structures in the external magnetic field is considered and the general form of the free energy and magnetization for arbitrary spin is derived. The exact relation between the free energies on infinite entire tree and on its infinite "interior" is obtained.

[*]e-mail: ananik@jerewan1.yerphi.am

[†]e-mail: izmailan@phys.sinica.edu.tw

[‡]Permanent address: Yerevan Physics Institute, Alikhanian Br.2, 375036 Yerevan, Armenia

[§]e-mail: kogan@hermes.lnf.infn.it

[¶]Permanent address: Yerevan Physics Institute, Alikhanian Br.2, 375036 Yerevan, Armenia

1. INTRODUCTION

Many theoretical problems of statistical mechanics, solid state physics, gauge models, polymers, etc., can be solved exactly on recursive structures, like the Bethe [1, 2, 3, 4] and the Husimi [5, 6, 7] lattices. Moreover, it has been argued recently [8] that, in some case, the Bethe lattice calculations are more reliable than mean-field calculations. In connection with this, the knowledge of the exact form of the free energy and magnetization is very actual for locating phase transitions and for analytical investigations of physical phenomenon. The main problem is how to calculate the free energy on the infinite "interior" of the infinite entire lattice. Namely, how get rid of the surface (with nonzero density of sites) contributions which can allow to strange results and rather different from those on regular lattices [9]. In our previous papers the relation between the free energies on Bethe and Cayley trees for spin-1 Ising model [3] and on Husimi and corresponded to it Cayley trees for multisite Ising model [7] are obtained. In that approach the free energy derived from the recursion relations and then the differentiation of the free energy functionals over the external magnetic field allowed to the exact relations. Recently, a method of computing the bulk free energy for any model on infinite "interior" tree is presented [8].

In this paper we have generalized the approach of the surface independent free energy calculation [3, 7] for an Ising spin - S model in an external magnetic field on generalized recursive lattice consisted of p - polygons (p is the number of edges of the polygon).

2. THE MODEL

The recursive tree consisted of p - polygons is characterized by p, the number of edges (the number of sites) of the polygon ($p = 2$ - usual Bethe lattice, $p = 3$ - Husimi lattice) and by q, the number of p-polygons which go out from each site (Fig.1).

These infinite dimensional recursive lattices cannot be embedded in any lattice belonging to an Euclidean space of finite dimension. However, they can be embedded in two-dimensional space of constant negative curvature (the hyperbolic or Lobachevsky plan **H2**) and study their metric properties [10].

Let us define the Ising spin-S model in an external magnetic field by Hamiltonian

$$- \beta H = \sum_{p-poligon} H_1(\sigma_i^{(1)}, \sigma_i^{(2)}, \cdots \sigma_i^{(p)}) + \sum_i H_2(\sigma_i), \tag{1}$$

where σ_i take values $(-S, -S + 1, ..., S - 1, S)$, $\sigma_i^{(k)}$ - spin around polygon, the first sum goes over all p - polygons and the second over all sites.. $H_1(\sigma_i^{(1)}, \sigma_i^{(2)}, ..., \sigma_i^{(p)})$ - includes all possible nearest - neighbor pair interactions, as well multisite interactions between spins belongs the same polygon and $H_2(\sigma_i) = \sum_{\mu=1}^{2S} h_\mu \sigma_i^\mu$ - includes all possible single - ion interactions.

The partition function will have the form

$$Z = \sum_{\{\sigma\}} \exp \left\{ \sum_{p-poligon} H_1(\sigma_i^{(1)}, \sigma_i^{(2)}, \cdots \sigma_i^{(p)}) + \sum_i H_2(\sigma_i) \right\}, \tag{2}$$

2

The advantage of the recursive tree is that for models formulated on it exact recursion relations can be derived. When this tree is cut apart at the central site 0, it separates into q identical branches, each of which contains $(q-1)(p-1)$ branches. Then the partition function can be written as follows:

$$Z = \sum_{\sigma_0} \exp\{H_2(\sigma_0)\} g_N^q(\sigma_0), \tag{3}$$

where σ_0 is a spin in central site, N is the number of generations, and $g_N(\sigma_0)$ is the partition function of a branch. Each branch, in turn, can be cut along any site of the 1th-generation which is nearest to the central site. The expression for $g_N(\sigma_0)$ can therefore be written in the form

$$g_N(\sigma_0) = \sum_{\{\sigma_1^{(j)}\}} \exp\left\{H_1(\sigma_0, \sigma_1^{(1)}, \cdots \sigma_1^{(p-1)}) + \sum_{i=1}^{p-1} H_2(\sigma_1^{(i)})\right\} g_{N-1}^{q-1}(\sigma_1^{(1)}) \cdots g_{N-1}^{q-1}(\sigma_1^{(p-1)}), \tag{4}$$

where $\sigma_1^{(j)}$ are the spins of first polygon except the central site. Consequently, we will have $2S+1$ recursion relations for the $g_N(\sigma_0)$, where σ_0 take values $(-S, -S+1, ..., S-1, S)$. After dividing each of the recursion relations on a recursion relation for $g_N(S)$, we will have $2S$ recursion relations for $x_N(\sigma_0)$

$$x_N(\sigma_0) = \frac{\sum_{\{\sigma_1^{(j)}\}} \exp\left\{H_1(\sigma_0, \sigma_1^{(1)}, \cdots \sigma_1^{(p-1)}) + \sum_{i=1}^{p-1} H_2(\sigma_1^{(i)})\right\} x_{N-1}^{q-1}(\sigma_1^{(1)}) \cdots x_{N-1}^{q-1}(\sigma_1^{(p-1)})}{\sum_{\{\sigma_1^{(j)}\}} \exp\left\{H_1(S, \sigma_1^{(1)}, \cdots \sigma_1^{(p-1)}) + \sum_{i=1}^{p-1} H_2(\sigma_1^{(i)})\right\} x_{N-1}^{q-1}(\sigma_1^{(1)}) \cdots x_{N-1}^{q-1}(\sigma_1^{(p-1)})}, \tag{5}$$

where

$$x_N(\sigma_0) = g_N(\sigma_0)/g_N(S), \tag{6}$$

and an equation for $g_N(S)$

$$g_N(S) = g_{N-1}^{(q-1)(p-1)}(S)\Psi_{N-1}, \tag{7}$$

where

$$\Psi_{N-1} = \sum_{\{\sigma_1^{(j)}\}} \exp\left\{H_1(S, \sigma_1^{(1)}, \cdots \sigma_1^{(p-1)}) + \sum_{i=1}^{p-1} H_2(\sigma_1^{(i)})\right\} x_{N-1}^{q-1}(\sigma_1^{(1)}) \cdots x_{N-1}^{q-1}(\sigma_1^{(p-1)}). \tag{8}$$

Since the right-hand side of Eq. (5) is bounded in x_N, it follows that x_N is finite for $N \to \infty$.

Through this $x_N(\sigma)$ one can express the density $m_\mu = \langle \sigma_0^\mu \rangle$ of central site (the symbol $\langle ... \rangle$ denotes the thermal average), where μ takes values from 1 to $2S$:

$$m_\mu = \langle \sigma_0^\mu \rangle = \frac{\sum_{\sigma_0} \sigma_0^\mu \exp\{H_2(\sigma_0)\} x_N^q(\sigma_0)}{\sum_{\sigma_0} \exp\{H_2(\sigma_0)\} x_N^q(\sigma_0)} \tag{9}$$

and other thermodynamic parameters. So we can say that the $x_N(\sigma)$ in the thermodynamic limit $(N \to \infty)$ determine the states of the system.

Using the Eq. (6), the free energy on the Cayley tree can be written as follows

$$-\beta F_N^{Cayley} = q \ln g_N(S) + \ln(\Phi_N), \tag{10}$$

3

where
$$\Phi_N = \sum_{\sigma_0} \exp H_2(\sigma_0) x_N^q(\sigma_0). \tag{11}$$

By substituting in the right-hand side of Eq. (10) the expression of $g_N(S)$ given by Eq. (7), we obtain the recursion relation for the free energy on the generalized Cayley tree

$$-\beta F_N^{Cayley} = -(q-1)(p-1)\beta F_{N-1}^{Cayley} + q \ln(\Psi_{N-1}) - (q-1)(p-1)\ln(\Phi_{N-1}) + \ln(\Phi_N) \tag{12}$$

By repeating this recursion procedure n times one can obtain

$$-\beta F_N^{Cayley} = -[(q-1)(p-1)]^n \beta F_{N-n}^{Cayley} - \beta F_{nN}, \tag{13}$$

where

$$-\beta F_{nN} = q \sum_{k=1}^{n} [(q-1)(p-1)]^{k-1} \ln(\Psi_{N-k}) - [(q-1)(p-1)]^n \ln(\Phi_{N-n}) + \ln(\Phi_N). \tag{14}$$

Therefore we obtain the general form of the free energy for Ising spin-S model on generalized Cayley tree

$$-\beta F_N^{Cayley} = q \sum_{k=0}^{N-1} [(q-1)(p-1)]^{N-k-1} \ln(\Psi_k) + \ln(\Phi_N) \tag{15}$$

where Ψ_k and Φ_N are given by Eqs. (8) and (11).

As mentioned in the Introduction the above obtained free energy gives rise to unusual behavior due to surface sites (in particular, it depends on surface conditions) [9]. To overcome this problem we here consider only local properties of sites deep within the graph (i.e. infinitely far from the boundary in the limit $N \to \infty$). Indeed, we will consider only contribution to Z_N from the Bethe lattice.

Let us consider the case, when the series of solution of recursion relations given by Eq. (5) converge to a stable point at $N \to \infty$. In this case $\lim x_{N-n}(\sigma) = x(\sigma)$ for all finite n and the recursion equations become

$$x(\sigma_0) = \frac{\sum_{\{\sigma_1^{(j)}\}} \exp \left\{ H_1(\sigma_0, \sigma_1^{(1)}, \cdots \sigma_1^{(p-1)}) + \sum_{i=1}^{p-1} H_2(\sigma_1^{(i)}) \right\} x^{q-1}(\sigma_1^{(1)}) \cdots x^{q-1}(\sigma_1^{(p-1)})}{\sum_{\{\sigma_1^{(j)}\}} \exp \left\{ H_1(S, \sigma_1^{(1)}, \cdots \sigma_1^{(p-1)}) + \sum_{i=1}^{p-1} H_2(\sigma_1^{(i)}) \right\} x^{q-1}(\sigma_1^{(1)}) \cdots x^{q-1}(\sigma_1^{(p-1)})}. \tag{16}$$

Then the Eq. (14) will take the form

$$-\beta F_n = \lim_{N \to \infty} (-\beta F_{nN})$$

$$= q \frac{(q-1)^n (p-1)^n - 1}{(q-1)(p-1) - 1} \ln(\Psi) - [(q-1)^n (p-1)^n - 1] \ln(\Phi), \tag{17}$$

where $\Psi = \lim_{N \to \infty} \Psi_{N-k}$ and $\Phi = \lim_{N \to \infty} \Phi_{N-k}$ for all finite k and

$$\Psi = \sum_{\{\sigma_1^{(j)}\}} \exp \left\{ H_1(S, \sigma_1^{(1)}, \cdots \sigma_1^{(p-1)}) + \sum_{i=1}^{p-1} H_2(\sigma_1^{(i)}) \right\} x^{q-1}(\sigma_1^{(1)}) \cdots x^{q-1}(\sigma_1^{(p-1)}), \tag{18}$$

4

$$\Phi = \sum_{\sigma_0} \exp H_2(\sigma_0) x^q(\sigma_0). \tag{19}$$

Let us now prove that Eq. (17) gives the free energy of the spin - S Ising model on the generalized Bethe lattice consist of n generations.

First, let calculate the total number of the sites of the generalized Bethe lattice consist of n generations. By definition, the Bethe lattice is the union of equivalent sites and therefore it is easy to get the relation between the numbers of bonds and sites. As from each site go out q bonds and each bond contains two sites then $N_{bond} = \frac{q}{2} N_{site}$. On the other hand, the number of bonds of the Cayley tree or the Bethe lattice consist of n generations is

$$N_{bond} = q + q(q-1) + q(q-1)^2 + \cdots + q(q-1)^{n-1} = q\frac{(q-1)^n - 1}{q-2}.$$

So, the total number of sites of the Bethe lattice consist of n generations is

$$N_{site} = \frac{2}{q} N_{bond} = 2\frac{(q-1)^n - 1}{q-2}.$$

Above we calculate the the number of sites in the case when $p = 2$. It is easy to generalize this result for arbitrary p. Note, only, that in case when $p > 2$ from each site go out $2q$ bonds and the relation between the numbers of bonds and sites will looks like

$$N_{bond} = \frac{2q}{2} N_{site} = qN_{site},$$

and the number of bonds for arbitrary p will have the following form

$$N_{bond} = qp + qp(q-1)(p-1) + \cdots + qp(q-1)^{n-1}(p-1)^{n-1} = qp\frac{(q-1)^n(p-1)^n - 1}{(q-1)(p-1) - 1}.$$

Thus, the total number of sites of the generalized Bethe lattice [1] consist of n generations is

$$N_{site}^{Bethe} = p\frac{(q-1)^n(p-1)^n - 1}{(q-1)(p-1) - 1}. \tag{20}$$

Now we can write the explicit expression for the free energy per site for the spin - S Ising model on the generalized Bethe lattice

$$-\beta f_{Bethe} = -\frac{\beta F_n}{N_{site}^{Bethe}} = \frac{q}{p} \ln(\Psi) - \frac{(q-1)(p-1) - 1}{p} \ln(\Phi), \tag{21}$$

---

[1]One can reach to the same result in a different way as well. Indeed, one of the ways to fulfill the equivalent of the sites of the Bethe lattice is to put a periodical boundary condition on Cayley tree. Then the number of sites of the last shell of Cayley tree will be equal to $q(q-1)^{n-1}(p-1)^{n-1}$, whereas the number of sites of the corresponding Bethe lattice is $(q-1)^{n-1}(p-1)^{n-1}$. One can see that the number of sites on the boundary shall of the generalized Bethe lattice is q times less then the corresponding number on the generalized Cayley lattice. Consequently

$$N_{site}^{Bethe} = 1 + q(p-1) + q(q-1)(p-1)^2 + \cdots + q(q-1)^{n-2}(p-1)^{n-1}$$

$$+(q-1)^{n-1}(p-1)^n = p\frac{(q-1)^n(p-1)^n - 1}{(q-1)(p-1) - 1}.$$

where Ψ and Φ are given by Eqs. (18) and (19) respectively.

We wish to mention here that in our earlier paper [3], we had noted that this functional gives the exact form of the free energy for spin-1/2 Ising model on the Bethe lattice (p=2) [11]. Then we checked the correctness of it for a spin-1 Ising model on the Bethe lattice (p=2) by differentiation over the external magnetic field. Recently, the correctness of that functional is confirmed for multisite antiferomagnetic spin-1/2 Ising model on the Husimi tree (p=3) as well [7].

Below an effort is made to prove that this statement is correct for arbitrary spin - S Ising model on the generalized Bethe lattice.

With this aim in view, let us differentiate this free energy functional with respect to the external field h_μ. Without loss of generality, we will consider, for simplicity, the case $p = 2$. For arbitrary p, the proof will be accomplished in the same way.

After a little algebra, we obtained

$$
\begin{aligned}
-\frac{\delta}{\delta h_\mu}(\beta f_{Bethe}) &= \frac{\sum_{\sigma_0} \sigma_0^\mu \exp\{H_2(\sigma_0)\} x^q(\sigma_0)}{\sum_{\sigma_0} \exp\{H_2(\sigma_0)\} x^q(\sigma_0)} \\
&+ \frac{q}{2} \frac{\sum_{\sigma_0} \exp\{H_2(\sigma_0)\} x^{q-1}(\sigma_0) \mathbf{T}(\sigma_0)}{\sum_{\sigma,\sigma_0} \exp\{H_1(S,\sigma) + H_2(\sigma) + H_2(\sigma_0)\} x^{q-1}(\sigma) x^{q-1}(\sigma_0)}.
\end{aligned} \tag{22}
$$

Here

$$
\mathbf{T}(\sigma_0) = \sum_\sigma \exp\{H_2(\sigma)\} x^{q-1}(\sigma)
$$

$$
\times \left\{ [\sigma^k + (q-1)\frac{x'(\sigma)}{x(\sigma)}][x(\sigma_0)\exp\{H_1(S,\sigma)\} - \exp\{H_1(\sigma_0,\sigma)\}] + x'(\sigma_0)\exp\{H_1(S,\sigma)\} \right\}, \tag{23}
$$

where

$$
x'(\sigma) \equiv \frac{\delta}{\delta h_\mu} x(\sigma).
$$

It is easy to check that

$$
\mathbf{T}(\sigma_0) = \frac{\delta}{\delta h_\mu} \left\{ \sum_\sigma \exp\{H_2(\sigma)\} x^{q-1}(\sigma) [x(\sigma_0)\exp\{H_1(S,\sigma)\} - \exp\{H_1(\sigma_0,\sigma)\}] \right\}. \tag{24}
$$

After substituting in the right-hand side of Eq. (24) the expression of $x(\sigma_0)$ (Eq. (16)), which in the case $p = 2$ is

$$
x(\sigma_0) = \frac{\sum_\sigma \exp\{H_1(\sigma_0,\sigma) + H_2(\sigma)\} x^{q-1}(\sigma)}{\sum_\sigma \exp\{H_1(S,\sigma) + H_2(\sigma)\} x^{q-1}(\sigma)}, \tag{25}
$$

we will obtain

$$
\mathbf{T}(\sigma_0) = 0
$$

and consequently

$$
-\frac{\delta}{\delta h_\mu}(\beta f_{Bethe}) = m_\mu = \frac{\sum_{\sigma_0} \sigma_0^\mu \exp\{H_2(\sigma_0)\} x^q(\sigma_0)}{\sum_{\sigma_0} \exp\{H_2(\sigma_0)\} x^q(\sigma_0)}. \tag{26}
$$

6

Thus, the expression given by Eq. (21) is the exact free energy functional per site for an Ising spin-S model on the generalized Bethe lattice. Consequently, the equation (Eq. (13)) [2] is the exact relation between the free energies on generalized Cayley tree and generalized Bethe lattice for a spin - S Ising model. It is obvious that this relation is model independent and depends only on the structure of recursive tree.

3. CONCLUSION

In this paper we have considered an Ising spin - S model on recursive p - polygonal structures in the external magnetic field and derive the general form of the free energy and magnetization for arbitrary spin. We have obtained the exact relation between the free energies on infinite entire tree and on its infinite "interior" and show that it is model independent and depends only on structure of recursive tree. The advantage of approach presented in this paper is that it gives possibility to obtain the exact expression for the surface independent free energy for width class of spin systems on generalized recursive lattices. This approach should be applicable for gauge models on generalized multi-plaquette recursive structures as well..

4. Acknowledgment

---

[2]More recently, an elegant geometrical interpretation of this relation is presented in Ref. [8]. The key idea of that method is the construction of a union trees with coordination number q from initial tree with the same coordination number.

References

[1] M. H. R. Tragtenberg, C. S. O. Yokoi, Phys. Rev. E **52** (1995) 2187; C. S. O. Yokoi, M. J. de Oliveira, and S. R. Salinas, Phys. Rev. Lett. **54** (1985) 163; S. Inawashiro, C. J. Thompson, C. Handa, J. Stat. Phys. **33** (1992) 419.

[2] P.D. Gujrati, J. Chem. Phys. **98** (1993) 1613; Phys. Rev. Lett. **53** (1984) 2453.

[3] N. S. Ananikian, A. R. Avakian and N. Sh. Izmailian, Physica A **172** (1991) 391.

[4] N. S. Ananikian and R. R. Shcherbakov, J. Phys. A **27** (1994) L887.

[5] J. L. Monroe, J. Stat. Physics **65** (1991) 255.

[6] N. S. Ananikian, R. R. Lusiniants, K. A. Oganessyan, JETP Lett. **61** (1995) 496; N. S. Ananikian, K. A. Oganessyan, Phys. Lett. A **200** (1995) 205.

[7] N. S.. Ananikian, S. K. Dallakian, N. Sh. Izmailian, and K. A. Oganessyan, Fractals **5** (1997) 175.

[8] P. D. Gujrati, Phys. Rev. Lett. **74** (1995) 809.

[9] L. K. Runnels, J. Math. Phys **8** (1967) 2081; T. P. Eggarter, J. Phys. C **7** (1974) 3433; E. Muller-Hartmann and J. Zittartz, Phys. Rev. Lett. **33** (1974) 893.

[10] J. A. de Miranda-Neto and F. Moraes, J. Phys. I France **3** (1993) 29; J. Physique - Letters **43** (1982) L-249.

[11] R. Baxter, *Exactly Solved Models in Statistical Mechanics*, Chapter **4**, New York, Academic Press, 1982.

5. Figure Caption

Fig.1 The recursive tree with $p = 4$ and $q = 2$. The numbers $0, 1, 2$ and 3 denote the central site, the sites of 1th, 2nd and 3rd generations, respectively.